The Poincare Unit Disc: An Interesting Example of an Abstract Riemann "Manifold" (Surface)

We have spoken already about how we might want to think about an "abstract" surface, that is (locally) a (u,v) domain with functions E, F, and G with $E>0$, $G>0$, and $EG-F^2>0$, so that $Q((a,b),(c,d)) \stackrel{\text{def}}{=} E\,ac + (ad+bc)F + G(bd)$ is a positive definite form. We think of this form as giving inner product of aS_u+bS_v with cS_u+dS_v, even though S_u, S_v are not given and maybe do not even exist! In this context, we get a natural idea of lengths of curves: If $\gamma(t) = (u(t), v(t))$ is a curve, we set (by definition)

$$\text{length of } \gamma = l(\gamma) = \int\left(\sqrt{E\left(\frac{du(t)}{dt}\right)^2 + 2F\frac{du(t)}{dt}\frac{dv(t)}{dt} + G\left(\frac{dv(t)}{dt}\right)^2}\right)dt$$

This would coincide with the usual idea when E, F, G come from a regular surface (patch) $S(u,v)$.

Notice that we can use our intrinsic formula for the Gauss curve in the $S(u,v)$ case as a definition in the abstract case! Also, using that $T(S_{uu}), T(S_{uv}), T(S_{vv})$ are intrinsic we can make a definition of geodesics on an abstract surface, too: We recall for this that a curve $\gamma(t) = S(u(t), v(t))$ is a geodesic if and only if

$$\frac{d^2u}{dt^2} S_u + \frac{d^2v}{dt^2} S_v + \left(\frac{du}{dt}\right)^2 T(S_{uu}) + 2\frac{du}{dt}\frac{dv}{dt} T(S_{uv}) + \left(\frac{dv}{dt}\right)^2 T(S_{vv}) = 0.$$

This vector equation is equivalent to two scalar equations (obtained by inner product with S_u & S_v)

$$\frac{d^2u}{dt^2} E + \frac{d^2v}{dt^2} F + \left(\frac{du}{dt}\right)^2 < T(S_{uu}), S_u > + 2\frac{du}{dt}\frac{dv}{dt} < T(S_{uv}), S_u > + \left(\frac{dv}{dt}\right)^2 < T(S_{vv}), S_u > = 0$$

and $$\frac{d^2u}{dt^2} F + \frac{d^2v}{dt^2} G + \left(\frac{du}{dt}\right)^2 < T(S_{uu}), S_v > + 2\frac{du}{dt}\frac{dv}{dt} < T(S_{uv}), S_v > + \left(\frac{dv}{dt}\right)^2 < T(S_{vv}), S_v > = 0.$$

Now
$$< T(S_{uu}), S_u > = \tfrac{1}{2} E_u$$
$$< T(S_{uv}), S_u > = \tfrac{1}{2} E_v$$
$$< T(S_{vv}), S_u > = F_v - \tfrac{1}{2} G_u$$
$$< T(S_{uu}), S_v > = F_u - \tfrac{1}{2} E_v$$
$$< T(S_{uv}), S_v > = \tfrac{1}{2} G_u$$
$$< T(S_{vv}), S_v > = \tfrac{1}{2} G_v$$
, as earlier.

So we can define two (geodesic) equations, conditions for $\gamma(t) = (u(t), v(t))$ to be a geodesic as follows:

$$\frac{d^2u}{dt^2} E + \frac{d^2v}{dt^2} F + \left(\frac{du}{dt}\right)^2 \left(\tfrac{1}{2} E_u\right) + 2\frac{du}{dt}\frac{dv}{dt}\left(\tfrac{1}{2} E_v\right) + \left(\frac{dv}{dt}\right)^2 \left(F_v - \tfrac{1}{2} G_u\right) = 0$$

and $$\frac{d^2u}{dt^2} F + \frac{d^2v}{dt^2} G + \left(\frac{du}{dt}\right)^2 \left(F_u - \tfrac{1}{2} E_v\right) + 2\frac{du}{dt}\frac{dv}{dt}\left(\tfrac{1}{2} G_u\right) + \left(\frac{dv}{dt}\right)^2 \left(\tfrac{1}{2} G_v\right) = 0$$

This will be the same as our original geodesic (vector) equation when E, F, G comes from a patch $S(u,v)$.

All of this is illustrated by an interesting and important example known as the Poincare unit disc. The (u,v) domain is $\{(u,v) \in \mathbb{R}^2 : u^2+v^2<1\}$ and

$E(u,v) = G(u,v) = 4/(1-u^2-v^2)^2$
$F(u,v) = 0$, called the "Poincare metric".

Let us change the parameterization here to polar coordinates (in the (u,v) plane), (r,θ). Note that the Poincare metric is at each point the Euclidean metric multiplied by the factor $4/(1-u^2-v^2)^2$. So in (u,v) polar coordinates the metric is $4/(1-r^2)^2$ times the euclidean metric, or, in usual calculus notation

$$\frac{4}{(1-r^2)^2}\left(dr^2 + r^2\, d\theta^2\right).$$

So the arclength of a curve $\gamma(t) = (r(t), \theta(t))$ is

$$\int \frac{2}{1-r^2}\left(\left(\frac{dr(t)}{dt}\right)^2 + r^2(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)^{\frac{1}{2}} dt$$

Let us compute the "Poincaré length" of the curve $\gamma(t) = (\underset{r(t)}{t}, \underset{\theta(t)}{0})$ $t \in [0,R]$. This =

$$\int_0^R \frac{2}{1-t^2}\left(1^2 + t^2 \cdot 0\right)\, dt$$

$$= \int_0^R \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt = \ln(1+R) - \ln(1-R)$$

$$= \ln\left(\frac{1+R}{1-R}\right).$$

In particular, as $R \to 1^-$, the length of γ on $[0,R]$ goes to $+\infty$: the boundary circle $R=1$ is "infinitely far away" from the origin in the Poincaré metric, at least along the straight line curve γ from the origin. But actually it is not hard to see that the boundary is infinitely far away along any curve: since

$$\sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2} \geq \left|\frac{dr}{dt}\right|,$$

one sees that varying θ as well as r just uses more arclength so that our straightline (θ constant) curve goes out to the boundary as efficiently as possible — but it is still infinite distance to the boundary. So any curve out to the boundary has infinite length. A bug living in the Poincaré disc feels that the universe is infinite! The curve γ with arclength parameter can be checked to be a geodesic.

Now we would like to figure out the Gauss curvature of the Poincare disc. Of course in principle we can compute this from the original E, F, G in (u, v) coordinates. But it is easier to look at the metric in (r, θ) coordinates. [Lurking behind the scenes here is a lengthy but automatic exercise showing that the Gauss curvature as intrinsically defined is independent of parameterization. For surface patches $S(u,v)$ we already know this by computation using the $(L_{11}L_{22} - L_{12}^2)/(EG-F^2)$ definition. But for the abstract case, we have to check directly that the intrinsic Gauss curvature formula is independent of parameterization. It would be peculiar if this did not work — and in fact it does work].

In (r, θ) coordinates

$$E = 4/(1-r^2)^2, F = 0, \quad G = 4r^2/(1-r^2)^2$$

Let us reparameterize again with (L, θ) where $L(r) = \ln((1+r)/(1-r))$.

Then it turns out that the new $E = 1$. This is automatic, since $L(r)$ = the Poincare length of the straight line from $(0,0)$ to the point with (euclidean) coordinates $(0, r)$, according to page 4.

We want to find G in terms not of r but of L (and θ, but there is no θ dependence). For this, we compute from $L = \ln((1+r)/(1-r))$ that $e^L = (1+r)/(1-r)$ so that $r = (e^L-1)/(e^L+1)$ and

$$\frac{4r^2}{(1-r^2)} = 4\left(\frac{e^L-1}{e^L+1}\right)^2 \Big/ \left(1 - \frac{(e^L-1)^2}{(e^L+1)^2}\right)^2$$

$$= 4\,(e^L-1)^2(e^L+1)^2 \big/ \left((e^L+1)^2 - (e^L-1)^2\right)^2$$

$$= \left[2\,(e^L-1)(e^L+1) \big/ (\cancel{e^{2L}} + 0)\right]^2 \quad +4e^{L}$$

$$= \left[\frac{e^{2L}-1}{2e^L}\right]^2 = \sinh^2 L.$$

(Note that change in θ is same vector in all coordinate systems here).

Thus in our new parameterization we have

$E = 1$, $F = 0$, $G = \sinh^2 L$

(L, θ as parameters).

Notice that this is the same E, F, G form as for a surface of revolution with (in our usual notation) $g(u) = \sinh(u)$. So by our previous results and the intrinsic nature of Gauss curvature, for $u > 0$, or $L > 0$,

Gauss curvature of Poincare metric

$$= -\left(\frac{d^2}{dL^2}\sinh L\right) \Big/ \sinh L = -1.$$

By continuity this is also true at $(0,0)$. The Poincare disc has Gauss curvature $\equiv -1$.

Tricky point: $(\sinh') = \cosh \geq 1$ so there is actually no literal surface of revolution with $g(u) = \sinh u$ (since $f' > 0$ and $(f')^2 + (g')^2 = 1$ for surf of rev). But the formal intrinsic calculation is still the same for the two cases.